Int J Theor Phys (2008) 47: 2021–2028
DOI 10.1007/s10773-007-9646-1

Canonical Entropy of Black Hole in the Generalized Uncertainty Principle

Li-Chun Zhang · Huai-Fan Li · Ren Zhao · Yue-Qin Wu

Received: 12 October 2007 / Accepted: 28 December 2007 / Published online: 15 January 2008

Abstract Recently, there has been much attention devoted to resolving the quantum corrections to the Bekenstein-Hawking black hole entropy. In particular, many researchers have expressed a vested interest in the coefficient of the logarithmic term of the black hole entropy correction term. In this paper, based on the correction to black hole thermodynamic quantity due to the generalized uncertainty principle, we calculate the partition function by energy spectrum obtained using tunneling effect. Furthermore we derive the black hole entropy. In the expression, we not only consider the generalized uncertainty principle but also consider the departure of black hole radiation spectrum from pure thermal spectrum. According to criterion law of thermodynamic systems phase transition, we discuss the phase transition of AdS black hole and derive that the phase transition of AdS black hole is a secondary one.

Keywords Generalized uncertainty principle · Tunneling effect · Area theorem · Phase transition

1 Introduction

One of the most remarkable achievements in gravitational physics was the realization that black holes have temperature and entropy [1–4]. There is a growing interest in the black hole entropy. Because entropy has statistical physics meaning in the thermodynamic system, it is related to the number of microstates of the system. However, in the Einstein's general theory of relativity, the black hole entropy is a pure geometry quantity. If we compare the black hole with the thermodynamic system, we will find an important difference. The black hole is a vacancy with strong gravitation. But the thermodynamic system is composed of atoms and molecules. Based on the microstructure of thermodynamic systems, we can explain thermodynamic property by statistic mechanics of its microcosmic elements. Whether the black

L.-C. Zhang · H.-F. Li · R. Zhao (✉) · Y.-Q. Wu
Department of Physics, Shanxi Datong University, Datong 037009, People's Republic of China
e-mail: zhaoren2969@yahoo.com.cn

hole has interior degrees of freedom corresponding the black hole entropy [5]? Let us suppose that the Bekenstein-Hawking entropy can be attributed a definite statistical meaning. Then how might one go about identifying these microstates and, even more optimistically, counting them [6]? This is a key problem to study the black hole entropy.

Recently, one of the focuses is the research for correction to black hole entropy. Various methods had success at calculating the correction value of the black hole entropy [6–14]. However, the precise value of the logarithmic term coefficient in the black hole entropy correction term is unknown. Since we discuss the black hole entropy, we need study the quantum effect of the black hole. When we discuss radiation particles or absorption ones, we should consider the uncertainty principle. However, corrected by string theory and non-commutative geometry, the Heisenberg relation is replaced by the generalized uncertainty principle [15–17]. There are many literatures to discuss the correction to the black hole entropy by the generalized uncertainty principle [6, 9, 14, 18–26].

The previous work that Hawking [3, 4] interpreted the quantum effect of black hole as event horizon emits thermal radiation spectral set an epochal milestone for black hole physics. The discovery not only solved the contradiction in black hole thermodynamics but also indicated the inherent contact among quantum dynamics, thermodynamics and gravity. One of the important subjects in black hole physics is studying thermal properties of various black holes. Hawking pointed that vacuum fluctuation near the surface of the black hole would produce virtual particle pair. When the virtual particles with negative energy come into black hole via tunneling effect, the energy of the black hole will decrease. At the same time, the particle with positive energy may thread out the gravitation region outside the black hole. Equivalently, the black hole radiates a particle. However, Hawking testifies that there is not any potential barrier in the tunneling.

Recently, Parikh and Wilczek [27] discussed Hawking radiation by tunneling effect. They thought that tunneling in the process of the particle radiation had not potential barrier before particles radiated. Potential barrier is produced by radiation particles itself. That is, during the process of tunneling effect creation, the energy of the black hole decrease and the radius of the black hole horizon reduce. The horizon radius becomes a new value that is smaller than the original value. The decrease of radius is determined by the value of energy of radiation particles. There is a classical forbidden band-potential barrier between original radius and the one after the black hole radiates. Parikh and Wilczek skillfully obtained the radiation spectrum of Schwarzschild and Reissner-Nordstrom black holes. Especially, their results are consistent with unitary principle in quantum mechanics. This supported the information conservation in Hawking radiation [27–29]. After this, in Parikh framework the primary result of Parikh has been generalized to many static and stationary spacetime [30–42]. Energy spectrum of any black hole radiation particles is

$$\rho_s \propto e^{\Delta S}, \tag{1}$$

where

$$\begin{aligned}\Delta S &= S_{MC}(E-E_s) - S_{MC}(E) = \sum_{k=1} \frac{1}{k!}\left(\frac{\partial^k S_{MC}(E_b)}{\partial E_b^k}\right)_{E_s=0} (-E_s)^k \\ &= -\beta E_s + \beta_2 E_s^2 + \cdots,\end{aligned} \tag{2}$$

$E_b = E - E_s$, β is the inverse of Hawking radiation temperature, E_s is the energy of radiation particles, $S_{MC}(E)$ is the microcanonical ensemble entropy with energy E,

$$\beta_k = \frac{1}{k!}\left(\frac{\partial^k \ln \Omega}{\partial E_b}\right)_{E_s=0} = \frac{1}{k!}\left(\frac{\partial^k S_{MC}}{\partial E_b}\right)_{E_s=0}. \tag{3}$$

Previous research of black hole entropy is based on the fact that the black hole has thermal radiation and radiation spectrum is a pure thermal spectrum. However, Hawking radiation derives the pure thermal spectrum is under the condition that spacetime is invariable. An obvious dispute is information loss. Black hole information loss means that pure quantum state will become a mixed state. This violates unitary principle in quantum mechanics. When we study black hole radiation by tunneling effect method, considering conversation of energy and change of horizon, we obtain that the radiation spectrum of the black hole is no longer a strict pure thermal spectrum. This method avoids the limitation of Hawking radiation and points out it is the gravitation that provide the potential barrier.

Using the radiation spectrum (1) obtained by tunneling effect method, we derive the correction to the black hole entropy. Our calculation method is different from the method of Majumdar [7, 10]. In the calculation of the partition function, we adopt probability integral method but Majumdar adopted saddle-point approximation. Using the relation between the partition function and entropy, we derive the canonical entropy of the black hole after considering the generalized uncertainty principle. In the correction term of the obtained entropy, there is not only logarithmic term of horizon area but also the term related to black hole thermal capacity. The term related to black hole thermal capacity is very important when we discuss the black hole phase transition. Based on this term, when we take the black hole as a general thermodynamics, we can judge phase transition of AdS black hole is primary or secondary.

2 Canonical Entropy of the Black Hole Considering the Generalized Uncertainty Principle

For the continuity of the paper, we recall the method of calculating canonical entropy of black hole in [43]. Taking the black hole as a general thermodynamic system, from (1), we can derive the partition function of the system is as follows:

$$Z(\beta) = \sum_s \rho_s. \tag{4}$$

Semiclassical partition function of canonical distribution can be expressed as

$$Z(\beta) = \int_0^\infty e^{\Delta S} dE \rho(E), \tag{5}$$

where, $\rho(E)$ is state density. $\rho(E) \equiv e^{S_{MC}(E)}$. Based on (2), when the energy of black hole radiation particles is E_s, black hole energy is $E_b = E - E_s$. Thus for the black hole, when the energy is E_b, the corresponding state density is $\rho(E - E_s)$. Thus

$$\rho(E - E_s) = \exp\left[S_{MC}(E - E_s)\right]. \tag{6}$$

After considering the generalized uncertainty principle, we expand micro-canonical entropy $S_{MC}(E-E_s)$ as Taylor series near energy E,

$$S_{MC}(E-E_s) = S_{MC}(E) - \beta E_s + \beta_2 E_s^2 + \cdots. \tag{7}$$

Ignoring higher order terms, (5) can be written as

$$\begin{aligned} Z(\beta) &= \int_0^\infty e^{-\beta E_s + \beta_2 E_s^2} dE_s e^{S_{MC}(E-E_s)} = e^{S_{MC}(E)} \int_0^\infty e^{-2\beta E_s + 2\beta_2 E_s^2} dE_s \\ &= e^{S_{MC}(E)} \left[\frac{1}{2}\sqrt{\frac{\pi}{-2\beta_2}} \exp\left(\frac{\beta^2}{-2\beta_2}\right) \left(1 - \operatorname{erf}\left(\frac{\beta}{\sqrt{-2\beta_2}}\right)\right)\right], \end{aligned} \tag{8}$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

is probability integral.

Using the relation between the partition function and entropy,

$$S = \ln Z - \beta \frac{\partial \ln Z}{\partial \beta}, \tag{9}$$

we can derive the entropy of the canonical ensemble

$$S(E) = S_{MC}(E) + \Delta, \tag{10}$$

where

$$\Delta = \ln f(\beta) - \beta \frac{\partial \ln f(\beta)}{\partial \beta}, \tag{11}$$

$$f(\beta) = \frac{1}{2}\sqrt{\frac{\pi}{-2\beta_2}} \exp\left(\frac{\beta^2}{-2\beta_2}\right) \left[1 - \operatorname{erf}\left(\frac{\beta}{\sqrt{-2\beta_2}}\right)\right]. \tag{12}$$

Thermal capacity of the system is

$$C \equiv -\beta^2 \left(\frac{\partial E}{\partial \beta}\right) \tag{13}$$

and

$$\beta_2 = -\frac{1}{2}\frac{\beta^2}{C}. \tag{14}$$

For probability integral, when $|z| < \infty$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{k=0}^{\infty} \frac{2^k z^{2k+1}}{(2k+1)!!}. \tag{15}$$

So

$$f(\beta) = \frac{1}{2}\sqrt{\frac{\pi}{-2\beta_2}} \exp\left(\frac{\beta^2}{-2\beta_2}\right) \left[1 - \frac{2}{\sqrt{\pi}} \exp\left(\frac{\beta^2}{2\beta_2}\right) \sum_{k=0}^{\infty} \frac{2^k}{(2k+1)!!} \left(\frac{\beta}{\sqrt{-2\beta_2}}\right)^{2k+1}\right]. \tag{16}$$

Substituting (14) and (16) into (11), we derive

$$\Delta = \frac{1}{2}\ln[CT^2] + \ln\left[1 - \frac{2}{\sqrt{\pi}}e^{-C}\sum_{k=0}^{\infty}\frac{2^k}{(2k+1)!!}\sqrt{C}^{2k+1}\right] - C$$
$$- \frac{2}{\sqrt{\pi}}e^{-C}\frac{\sum_{k=0}^{\infty}[\frac{2^k}{(2k+1)!!}\sqrt{C}^{2k+1}((2k+1)-2C)]}{1-\frac{2}{\sqrt{\pi}}e^{-C}\sum_{k=0}^{\infty}\frac{2^k}{(2k+1)!!}\sqrt{C}^{2k+1}} + const., \tag{17}$$

where $T = 1/\beta$ is system temperature. When $|C| \ll 1$,

$$\Delta \approx \ln CT^2. \tag{18}$$

Near critical point, when $C \to \infty$, from (14), $|z| \to \infty$. According to the approximately expression of probability integral

$$\mathrm{erf}(z) = 1 - \frac{e^{-z^2}}{\sqrt{\pi}z}\left[1 + \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{(2z^2)^k}\right], \quad |z| \to \infty,$$

we obtain

$$f(\beta) = \frac{1}{2\beta}\left[1 + \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k}\left(\frac{\sqrt{-2\beta_2}}{\beta}\right)^{2k}\right]. \tag{19}$$

Substituting (19) and (14) into (11), we can obtain the correction to entropy near critical point,

$$\Delta = \ln\left[\frac{1}{2\beta} + \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k 2\beta}\left(\frac{\sqrt{-2\beta_2}}{\beta}\right)^{2k}\right]$$
$$+ \frac{1+\sum_{k=1}^{\infty}(-1)^k(2\sqrt{-2\beta_2})^{2k}\frac{(2k+1)(2k-1)!!}{2^k(2\beta)^{2k}}}{1+\sum_{k=1}^{\infty}(-1)^k(2\sqrt{-2\beta_2})^{2k}\frac{(2k-1)!!}{2^k(2\beta)^{2k}}}$$
$$= \ln T\left[1 + \sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k C^k}\right] + \frac{1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k+1)(2k-1)!!}{2^k C^k}}{1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k C^k}}. \tag{20}$$

3 Application Example

From [6], considering the generalized uncertainty principle, micro-canonical entropy of Schwarzschild black hole is given by

$$S_{MC}(E) = \frac{A}{4} - \frac{\pi\alpha^2}{4}\ln\left(\frac{A}{4}\right) + \sum_{n=1}^{\infty}C_n\left(\frac{A}{4}\right)^{-n} + const., \tag{21}$$

where α is a dimensionless constant of order one, $A = 16\pi M^2$ is the horizon area of the black hole. Based on the black hole thermal capacity given by (21)

$$C = T\left(\frac{\partial S_{MC}}{\partial T}\right) = \frac{\alpha^2}{16} + \sum_{n=1}^{\infty}2nC_n\left(\frac{A}{4}\right)^{-n} - 8\pi M^2. \tag{22}$$

When the energy of the black hole M satisfies

$$\frac{\alpha^2}{16}+\sum_{n=1}^{\infty}2nC_n\left(\frac{A}{4}\right)^{-n}>8\pi M^2 \tag{23}$$

the thermal capacity is positive, the expression of the black hole entropy is

$$S=S_{BH}-\frac{\pi\alpha^2}{4}\ln S_{BH}+\sum_{n=1}^{\infty}C_c S_{BH}^{-n}+\ln CT+const., \tag{24}$$

where S_{BH} is Bekenstein-Hawking entropy. After considering the generalized uncertainty principle, from (23), Schwarzschild black hole can be in the thermodynamic stable state.

The metric of a SAdS black hole in five-dimensional spacetime is given by

$$ds^2=-\left(1-\frac{16\pi G_5 M}{3\Omega_3 r^2}+\frac{r^2}{l^2}\right)dt^2+\left(1-\frac{16\pi G_5 M}{3\Omega_3 r^2}+\frac{r^2}{l^2}\right)^{-1}dr^2+r^2 d\Omega_3^2, \tag{25}$$

where Ω_3 is the metric of the unit S^3 and G_5 is the five-dimensional Newton's constant. The Bekenstein-Hawking entropy and Hawking temperature is

$$S_{BH}=\frac{\Omega_3 r_+^3}{4G_5},\qquad T=\frac{4r_+^2+2l^2}{4\pi l^2 r_+}, \tag{26}$$

where r_+ satisfies $1-\frac{16\pi G_5}{3\Omega_3 r_+^2}+\frac{r_+^2}{l^2}=0$.

After considering the generalized uncertainty principle, micro-canonical entropy of the black hole [14]

$$S_{MC}=S_{BH}\left(1-\frac{3\alpha^2}{4r_+^2}\right). \tag{27}$$

Thermal capacity of the black hole

$$C=\frac{3\Omega_3 r_+(r_+^2-\alpha^2/4)(2r_+^2+l^2)}{4(2r_+^2-l^2)G_5}=3\frac{2r_+^2+l^2}{2r_+^2-l^2}\left[S_{BH}-\alpha^2\frac{\Omega_3}{16}r_+\right]. \tag{28}$$

When $|C|<\infty$, that is $r_+\gg l$, the expression of the black hole entropy is

$$S=S_{BH}\left(1-\frac{3\alpha^2}{4r_+^2}\right)+\ln CT+const. \tag{29}$$

when $|C|\to\infty$, that is $2r_+^2\to l^2$, the black hole entropy near the critical point can be written as

$$S=S_{BH}\left(1-\frac{3\alpha^2}{4r_+^2}\right)+\ln T\left[1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k C^k}\right]+\frac{1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k+1)(2k-1)!!}{2^k C^k}}{1+\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)!!}{2^k C^k}}. \tag{30}$$

From (30), we obtain that when the black hole thermal capacity is divergent, the entropy is not divergent. According to the criterion law of thermodynamic system phase transition, we know that the phase transition of the black hole is the secondary. It is different from the result given by [44].

4 Conclusion and Discussion

Black hole thermodynamics foundation and the proof of Hawking radiation are the reason to believe that black hole is a thermodynamic system. According to the laws of general thermodynamic system, we can derive four laws of black hole thermodynamics. It is well known that the entropy of general thermodynamic systems is described by statistical mechanics. As black hole is a thermodynamic system, we are certainly interested in its statistical mechanics background. In recent years, string theory and loop quantum gravity both had success at statistically explaining the black hole entropy. However, which one is perfect? This is an open problem. Since black hole is a thermodynamic system, how can it demonstrate the characteristic phase transition?

In other hand, previous research of black hole entropy is based on the fact that the black hole has thermal radiation and radiation spectrum is a pure thermal spectrum. However, Hawking radiation derived the pure thermal spectrum is under the condition that spacetimes is invariable. An obvious dispute is information loss. Black hole information loss means that pure quantum state will become a mixed state. This violates unitary principle in quantum mechanics. When we study black hole radiation by tunneling effect method, considering conversation of energy and change of horizon, we obtain that the radiation spectrum of the black hole is no longer a strict pure thermal spectrum. This method avoids the limitation of Hawking radiation and points out it is the gravitation that provide the potential barrier.

According to the affect on black hole thermodynamic quantity due to the generalized uncertainty principle and applying the radiation spectrum obtained with quantum tunneling effect method, we calculate the partition function. In calculation, we adopt probability integral method. Using the relation between the partition function of general thermodynamic system and entropy, we derive the expression of the black hole canonical entropy. For Schwarzschild black hole, when the energy satisfies (23), the black hole can be in the thermodynamic stable state. For SAdS black hole, when thermal capacity $|C| \to \infty$, according to the criterion law of thermodynamic system phase transition, we know that the phase transition of the black hole is the secondary.

Acknowledgement This project was supported by the Shanxi Natural Science Foundation of China under Grant No. 2006011012.

References

1. Bekenstein, J.D.: Phys. Rev. D **7**, 2333 (1973)
2. Bekenstein, J.D.: Phys. Rev. D **9**, 3292 (1974)
3. Hawking, S.W.: Nature **248**, 30 (1974)
4. Hawking, S.W.: Commun. Math. Phys. **43**, 199 (1975)
5. Wang, Y.J.: Black Hole Physics, p. 263. Hunan Normal University Press, Changsha (2000) (in Chinese)
6. Medved, A.J.M., Vagenas, E.C.: Phys. Rev. D **70**, 124021 (2004)
7. Chatterjee, A., Majumdar, P.: Phys. Rev. Lett. **92**, 141301 (2004)
8. Kaul, R.K., Majumdar, P.: Phys. Rev. Lett. **84**, 5255 (2000)
9. Camellia, G.A., Arzano, M., Procaccini, A.: Phys. Rev. D **70**, 107501 (2004)
10. Chatterjee, A., Majumdar, P.: Phys. Rev. D **71**, 024003 (2005)
11. Myung, Y.S.: Phys. Lett. B **579**, 205 (2004)
12. Akbar, M.M., Das, S.: Class. Quantum Gravity **21**, 1383 (2004)
13. Das, S., Majumdar, P., Bhaduri, R.K.: Class. Quantum Gravity **19**, 2355 (2002)
14. Setare, M.R.: Int. J. Mod. Phys. A **2**, 11325 (2006)
15. Chang, L.N., Minic, D., Okaruma, N., Takeuchi, T.: Phys. Rev. D **65**, 125028 (2002)
16. Kempt, A., Mangano, G., Pobert, B.M.: Phys. Rev. D **52**, 1108 (1995)
17. Garay, L.J.: Int. J. Mod. Phys. A **10**, 145 (1995)

18. Kim, W., Kim, Y.W., Park, Y.J.: Phys. Rev. D **74**, 104001 (2006)
19. Kim, W., Kim, Y.W., Park, Y.J.: Phys. Rev. D **75**, 127501 (2007)
20. Setare, M.R.: Eur. Phys. J. C **33**, 555 (2004)
21. Setare, M.R.: Phys. Rev. D **70**, 087501 (2004)
22. Nouicer, K.: Phys. Lett. B **646**, 63 (2007)
23. Zhao, R., Zhang, S.L.: Phys. Lett. B **641**, 208 (2006)
24. Zhao, R., Wu, Y.Q., Zhang, L.C.: Class. Quantum Gravity **20**, 4885 (2003)
25. Li, X.: Phys. Lett. B **540**, 9 (2002)
26. Kim, W., Kim, Y.W., Park, Y.J.: Phys. Lett. B **645**, 393 (2007)
27. Parikh, M.K., Wilczek, F.: Phys. Rev. Lett. **85**, 5042 (2000)
28. Parikh, M.K.: Gen. Relativ. Gravit. **36**, 2419 (2004)
29. Parikh, M.K.: Phys. Lett. B **546**, 189 (2002)
30. Liu, C.Z., Zhang, J.Y., Zhao, Z.: Phys. Lett. B **639**, 670 (2006)
31. Zhang, J.Y., Zhao, Z.: Phys. Lett. B **638**, 110 (2006)
32. Zhang, J.Y., Zhao, Z.: Phys. Lett. B **618**, 14 (2005)
33. Zhang, J.Y., Zhao, Z.: Nucl. Phys. B **725**, 173 (2005)
34. Zhang, J.Y., Zhao, Z.: J. High Energy Phys. **10**, 055 (2005)
35. Li, H.L., Yang, S.Z., Jiang, Q.Q., Qi, D.J.: Phys. Lett. B **641**, 139 (2006)
36. Kerner, R., Mann, R.B.: Phys. Rev. D **73**, 104010 (2006)
37. Angheben, M., Nadalini, M., Vanzo, L., Zerbini, S.: J. High Energy Phys. **05**, 014 (2005)
38. Arzano, M., Medved, A.J.M., Vagenas, E.C.: J. High Energy Phys. **09**, 037 (2005)
39. Medved, A.J.M., Vagenas, E.C.: Mod. Phys. Lett. A **20**, 2449 (2005)
40. Medved, A.J.M., Vagenas, E.C.: Mod. Phys. Lett. A **20**, 1723 (2005)
41. Arzano, M.: Mod. Phys. Lett. A **21**, 41 (2006)
42. Zhao, R., Zhang, L.C., Hu, S.Q.: Acta Phys. Sin. **55**, 3898 (2006) (in Chinese)
43. Zhao, R., Zhang, L.C., Zhang, S.L.: Int. J. Theor. Phys. **46**, 2158 (2007)
44. Majumdar, P.: Class. Quantum Gravity **24**, 1747 (2007)